Inverse-scattering approach to femtosecond solitons in monomode optical fibers

D. Mihalache* and L. Torner
Department of Signal Theory and Communications, Polytechnic University of Catalonia, P.O. Box 30002, 08080 Barcelona, Spain

F. Moldoveanu and N.-C. Panoiu
Department of Theoretical Physics, Institute of Atomic Physics, P.O. Box MG-6, Bucharest, Romania

N. Truta
Department of Physics, Institute of Civil Engineering, Bucharest, Romania

(Received 30 March 1993)

Using the inverse-scattering transform with 3×3 U-V matrix representation and fully exploiting the symmetry properties of the scattering matrix elements, we found the one-parameter single-soliton, the four-parameter breather soliton, and the general N-soliton solutions of a perturbed nonlinear Schrödinger equation which describes the femtosecond pulse propagation in optical fibers. The threshold power below which the one-parameter single soliton cannot be formed was given. The main characteristic of the general single-soliton solution of the perturbed nonlinear Schrödinger equation is that it presents an arbitrary number of "humps" (local maxima of the amplitude) of different heights.

PACS number(s): 42.25.Bs, 42.50.Rh, 42.81.Dp

I. INTRODUCTION

Optical solitons in monomode optical fibers are pulses which propagate without any change in pulse shape or intensity. Because of their remarkable stability properties, optical solitons are now at the center of an active research field of nonlinear wave propagation in optical fibers. This research field started with the result obtained by Hasegawa and Tappert [1], which show that, under an appropriate combination of pulse shape and intensity, the effects of the intensity-dependent refractive index of the fiber exactly compensate for the pulse-spreading effects of the group-velocity dispersion. For the negative group-velocity-dispersion or anomalous dispersion regime, the fundamental soliton is called a bright pulse, and the propagation of these bright solitons has been studied intensively and verified experimentally [2]. For the positive group-velocity-dispersion or normal dispersion regime, the theory [1] and numerical simulations [3] predict that the solitons are dark pulses (i.e., a dip occurs at the center of the pulse). The generation of dark solitons in single-mode optical fibers was also demonstrated [4]. We mention also the works of several very active research groups in the field of pulse propagation in optical fibers in both the picosecond and femtosecond regimes [5–25].

The propagation of optical solitons in the ps domain can be well described by the nonlinear Schrödinger equation (NLSE) [1]. The NLSE is one of the completely integrable nonlinear partial differential equations and its solutions may be obtained by different methods, e.g., by using the inverse-scattering transform (IST) [26–30], the Lie group theory [31], by constructing a certain completely integrable finite-dimensional dynamical system whose solutions determine the exact solutions of the NLSE [32–34], etc. We mention also the recent work on IST perturbation theory for soliton propagation and the first- and the second-order perturbation expansion for soliton propagation in optical fibers [35].

The propagation of *fs* optical pulses in monomode optical fibers is well described by the following modified NLSE:

$$i\frac{\partial q}{\partial Z}+\frac{1}{2}\frac{\partial^2 q}{\partial T^2}+|q|^2q+i\left[\beta_1\frac{\partial^3 q}{\partial T^3}+\beta_2|q|^2\frac{\partial q}{\partial T}+\beta_3 q\frac{\partial|q|^2}{\partial T}\right]=\epsilon\left[-i\Gamma q+\sigma q\frac{\partial|q|^2}{\partial T}\right], \tag{1.1}$$

where q represents a normalized complex amplitude of the pulse envelope, Z is a normalized distance along the fiber, T is the normalized retarded time (we employ a frame of reference moving with the pulse with its group velocity), ϵ is a small parameter, β_1, β_2, β_3, Γ, and σ are real normalized parameters which depend on the fiber characteristics [23,24,36]. The last two terms in Eq. (1.1) describe the fiber loss effect and the self-induced Raman scattering effect. The very last term produces a shift of the central frequency of a soliton to a lower frequency when the soliton width is less then 1 ps [10]. We mention that in the past years many attempts have been made to find the solitary-wave solutions of the modified NLSE under various degrees of approximations [37–45].

To the best of our knowledge for arbitrary values of the parameters β_1, β_2, and β_3, Eq. (1.1) with $\epsilon=0$ is not

1063-651X/93/48(6)/4699(11)/$06.00

completely integrable. However, for an appropriate choice of these parameters it can be integrated by the IST. Thus the cases when $\beta_1{:}\beta_2{:}\beta_3=0{:}1{:}1$ (the derivative NLSE type I), $\beta_1{:}\beta_2{:}\beta_3=0{:}1{:}0$ (the derivative NLSE type II) and $\beta_1{:}\beta_2{:}\beta_3=1{:}6{:}0$ (the Hirota equation) were solved in [46–48], respectively. Recently, Sasa and Satsuma [49] showed that the case $\beta_1{:}\beta_2{:}\beta_3=1{:}6{:}3$ is also integrable by using the IST. The Sasa-Satsuma single-soliton solution may have either one "hump" or two "humps" of equal heights. Nevertheless, the authors of Ref. [49] obtained a particular single-soliton solution of Eq. (1.1) with $\beta_1{:}\beta_2{:}\beta_3=1{:}6{:}3$ and $\epsilon=0$.

In the present paper we find the most general single-soliton solution of the perturbed NLSE:

$$i\frac{\partial q}{\partial Z}+\frac{1}{2}\frac{\partial^2 q}{\partial T^2}+|q|^2q+i\delta\left[\frac{\partial^3 q}{\partial T^3}+6|q|^2\frac{\partial q}{\partial T}+3q\frac{\partial |q|^2}{\partial T}\right]=0 \quad (1.2)$$

classified by the following criteria: (i) the diagonal element of the scattering matrix $\alpha_{33}(\zeta)$ has only one zero on the imaginary axis; (ii) the diagonal element of the scattering matrix $\alpha_{33}(\zeta)$ has two zeros located symmetrically with respect to the imaginary axis.

With a specific choice of the parameters which describe the general single-soliton solution we also find the breather soliton solution for Eq. (1.2).

The paper is organized as follows. In Sec. II we present in detail the method to integrate Eq. (1.2) by using the IST. By taking into account the symmetry properties of the matrix U in the U-V representation for Eq. (1.2) we establish the symmetry properties of the Jost functions and of the scattering matrix elements α_{ij}. Using these symmetry relations we derive the corresponding Gel'fand-Levitan-Marchenko (GLM) equations and the time-dependence of the scattering data.

After that, in Sec. III we find the general single-soliton solution of Eq. (1.2) and we obtain as particular cases the one-parameter single soliton, and the four-parameter breather soliton solution. We show that by an appropriate choice of the soliton parameters one can obtain single-soliton solutions with an arbitrary number of "humps" of different heights. At the end of this section we briefly discuss the procedure to construct the general N-soliton solution.

Finally, in the last section we briefly summarize our conclusions.

In Appendix A we analyze the analyticity properties of the Jost functions and of the scattering matrix elements and in Appendix B we show in detail the procedure to obtain the integrals of motion.

II. THE INVERSE SCATTERING TRANSFORM

In order to integrate Eq. (1.2) we make, as in [49], the following transformation:

$$u(x,t)=q(T,Z)\exp\left[-\frac{i}{6\delta}\left[T-\frac{Z}{18\delta}\right]\right] \quad (2.1)$$

with $t=Z$ and $x=T-Z/12\delta$.

Thus Eq. (1.2) transforms to a complex modified Korteweg–de Vries (KdV) -type equation:

$$\frac{\partial u}{\partial t}+\delta\left[\frac{\partial^3 u}{\partial x^3}+6|u|^2\frac{\partial u}{\partial x}+3u\frac{\partial |u|^2}{\partial x}\right]=0\ . \quad (2.2)$$

In order to integrate Eq. (2.2) by IST we consider as in [49] the following eigenvalue problem:

$$\frac{\partial \Psi}{\partial x}=U\Psi\ , \quad (2.3)$$

where

$$U=\begin{bmatrix}-i\zeta & 0 & u\\ 0 & -i\zeta & u^*\\ -u^* & -u & i\zeta\end{bmatrix}, \quad (2.4)$$

Ψ is a column vector $(\Psi_1,\Psi_2,\Psi_3)^t$, and ζ is a time-independent spectral parameter. With the time evolution of the eigenvector Ψ given by

$$\frac{\partial \Psi}{\partial t}=V\Psi\ , \quad (2.5)$$

where

$$V=-4i\delta\zeta^3\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}+4\delta(\zeta^2-|u|^2)\begin{bmatrix}0 & 0 & u\\ 0 & 0 & u^*\\ -u^* & -u & 0\end{bmatrix}$$
$$+2i\delta\zeta\begin{bmatrix}|u|^2 & u^2 & u_x\\ u^{*2} & |u|^2 & u_x^*\\ u_x^* & u_x & -2|u|^2\end{bmatrix}-\delta\begin{bmatrix}0 & 0 & u_{xx}\\ 0 & 0 & u_{xx}^*\\ -u_{xx}^* & -u_{xx} & 0\end{bmatrix}+\delta(uu_x^*-u_xu^*)\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}, \quad (2.6)$$

the compatibility condition of Eqs. (2.3) and (2.5) is equivalent with Eq. (2.2). We note that IST with 3×3 U-V matrix representation has been discussed in [50–52].

Next for a real eigenvalue ζ we introduce the Jost functions $\varphi^{(i)}(x;\zeta)$ and $\psi^{(i)}(x;\zeta)$, $i=1,2,3$, which satisfy the following asymptotic conditions:

$$\varphi_j^{(i)}(x;\zeta)\to\delta_{ij}e^{i\gamma_i\zeta x},\quad x\to-\infty\ , \quad (2.7)$$

$$\psi_j^{(i)}(x;\zeta)\to\delta_{ij}e^{i\gamma_i\zeta x},\quad x\to\infty\ , \quad (2.8)$$

where $\gamma_1=\gamma_2=-1$ and $\gamma_3=1$. Because for real ζ the matrix U is anti-Hermitian ($U^+=-U$), we have

$$\frac{\partial}{\partial x}\left[\Psi^{(1)^+}\Psi^{(2)}\right]=0 \tag{2.9}$$

for any pair of solutions of Eq. (2.3) corresponding to the same eigenvalue ζ. Now we introduce the scattering matrix $\alpha=[\alpha_{ij}(\zeta)]_{i,j=1,2,3}$ via the following relationship:

$$\varphi^{(i)}(x;\zeta)=\sum_{j=1}^{3}\alpha_{ij}(\zeta)\psi^{(j)}(x;\zeta) \tag{2.10}$$

between the two bases $\{\varphi^{(i)}(x;\zeta)\}_{i=1,2,3}$ and $\{\psi^{(i)}(x;\zeta)\}_{i=1,2,3}$ in the space of solutions of Eq. (2.3). Because the matrix α is unimodular ($\det\alpha=1$) and taking into account that the two bases $\{\varphi^{(i)}(x;\zeta)\}_{i=1,2,3}$ and $\{\psi^{(i)}(x;\zeta)\}_{i=1,2,3}$ are orthogonal, we obtain that the matrix α is unitary, i.e., $\alpha^+=\alpha^{-1}$. Using this property and Eq. (2.10) one can easily find the following equations:

$$[\alpha_{22}(\zeta)\varphi^{(1)}e^{i\zeta x}-\alpha_{12}(\zeta)\varphi^{(2)}e^{i\zeta x}]/\alpha_{33}^*(\zeta)=\psi^{(1)}e^{i\zeta x}-[\alpha_{31}^*(\zeta)/\alpha_{33}^*(\zeta)]\psi^{(3)}e^{i\zeta x}\ , \tag{2.11a}$$

$$[-\alpha_{21}(\zeta)\varphi^{(1)}e^{i\zeta x}+\alpha_{11}(\zeta)\varphi^{(2)}e^{i\zeta x}]/\alpha_{33}^*(\zeta)=\psi^{(2)}e^{i\zeta x}-[\alpha_{32}^*(\zeta)/\alpha_{33}^*(\zeta)]\psi^{(3)}e^{i\zeta x}\ , \tag{2.11b}$$

$$\varphi^{(3)}e^{-i\zeta x}/\alpha_{33}(\zeta)=\psi^{(3)}e^{-i\zeta x}+[\alpha_{31}(\zeta)/\alpha_{33}(\zeta)]\psi^{(1)}e^{-i\zeta x}+[\alpha_{32}(\zeta)/\alpha_{33}(\zeta)]\psi^{(2)}e^{-i\zeta x}\ . \tag{2.11c}$$

In Appendix A the analyticity properties of the Jost functions and of the scattering matrix elements as functions of the spectral parameter ζ are studied.

Now we establish the symmetry properties of the Jost functions and of the scattering matrix α which will help us to find the most general single-soliton solution.

It is easy to observe that if $\Psi(x;\zeta)$ is a solution of the system (2.3) then $\tilde{\Psi}(x;\zeta)\equiv S\Psi^*(x;-\zeta^*)$ is also a solution of the system (2.3) where S is the following unitary, Hermitian matrix:

$$S=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}. \tag{2.12}$$

From the asymptotic behavior of the Jost functions $\varphi^{(i)}(x;\zeta)$ and $\psi^{(i)}(x;\zeta)$, $i=1,2,3$, one can find the subsequent relations:

$$\tilde{\varphi}^{(1)}(x;\zeta)=\varphi^{(2)}(x;\zeta)\ ,\quad \tilde{\varphi}^{(2)}(x;\zeta)=\varphi^{(1)}(x;\zeta)\ ,\quad \tilde{\varphi}^{(3)}(x;\zeta)=\varphi^{(3)}(x;\zeta)\ , \tag{2.13a}$$

$$\tilde{\psi}^{(1)}(x;\zeta)=\psi^{(2)}(x;\zeta)\ ,\quad \tilde{\psi}^{(2)}(x;\zeta)=\psi^{(1)}(x;\zeta)\ ,\quad \tilde{\psi}^{(3)}(x;\zeta)=\psi^{(3)}(x;\zeta)\ . \tag{2.13b}$$

Due to all these properties, one can find the following symmetry relations:

$$\alpha_{11}(\zeta)=\alpha_{22}^*(-\zeta^*),\quad \alpha_{12}(\zeta)=\alpha_{21}^*(-\zeta^*)\ ,\quad \alpha_{33}(\zeta)=\alpha_{33}^*(-\zeta^*),\quad \alpha_{31}(\zeta)=\alpha_{32}^*(-\zeta^*)\ ,\quad \alpha_{13}(\zeta)=\alpha_{23}^*(-\zeta^*)\ , \tag{2.14}$$

Next we derive the GLM equations. To this aim we introduce the integral representations of the Jost functions:

$$\psi_j^{(i)}(x;\zeta)=\delta_{ij}e^{i\gamma_i\zeta x}+\int_x^{\infty}ds\,K_j^{(i)}(x,s)e^{i\gamma_i\zeta s}\ , \tag{2.15}$$

where $K^{(i)}(x,s)=(K_1^{(i)}(x,s),K_2^{(i)}(x,s),K_3^{(i)}(x,s))^t$ with $\lim_{s\to\infty}K^{(i)}(x,s)=0$, $i=1,2,3$.

A direct consequence of the symmetry relations (2.14) is the property that the zeros of $\alpha_{33}(\zeta)$ are either on the imaginary axis in the lower complex half plane or located symmetrically with respect to the imaginary axis at $(\zeta^*,-\zeta)$, $\mathrm{Im}\zeta>0$. In the following we assume that $\alpha_{33}(\zeta)$ has N pairs of simple zeros located symmetrically with respect to the imaginary axis at $(\zeta_i^*,-\zeta_i)$, $\mathrm{Im}\zeta_i>0$, $i=1,2,\ldots,N$.

For $\zeta=\zeta_i^*$ we have

$$\varphi^{(3)}(x;\zeta_i^*)=c_{31}^{(i)}\psi^{(1)}(x;\zeta_i^*)+c_{32}^{(i)}\psi^{(2)}(x;\zeta_i^*) \tag{2.16a}$$

and for $\zeta=-\zeta_i$, by using the symmetry relations (2.14) we obtain

$$\varphi^{(3)}(x;-\zeta_i)=c_{32}^{(i)*}\psi^{(1)}(x;-\zeta_i)+c_{31}^{(i)*}\psi^{(2)}(x;-\zeta_i)\ . \tag{2.16b}$$

In a slightly different manner than in [49], by taking into account the symmetry relations (2.14) we find from Eqs. (2.11a)–(2.11c) the following GLM equations:

$$K^{(1)}(x,y)-\begin{bmatrix}0\\0\\1\end{bmatrix}F^*(x+y)-\int_x^{\infty}ds\,K^{(3)}(x,s)F^*(s+y)=0\ , \tag{2.17a}$$

$$K^{(2)}(x,y)-\begin{bmatrix}0\\0\\1\end{bmatrix}F(x+y)-\int_x^{\infty}ds\,K^{(3)}(x,s)F(s+y)=0\ , \tag{2.17b}$$

$$K^{(3)}(x,y)+\begin{bmatrix}1\\0\\0\end{bmatrix}F(x+y)+\begin{bmatrix}0\\1\\0\end{bmatrix}F^*(x+y)+\int_x^\infty ds\, K^{(1)}(x,s)F(s+y)+\int_x^\infty ds\, K^{(2)}(x,s)F^*(s+y)=0\ , \tag{2.17c}$$

respectively, for $y>x$, where $F(z)$ is given by

$$F(z)=\sum_{j=1}^{N} i\left[\frac{c_{31}^{(j)}}{\alpha_{33}'(\zeta_j^*)}\exp(-i\zeta_j^* z)+\frac{c_{32}^{(j)*}}{\alpha_{33}'(-\zeta_j)}\exp(i\zeta_j z)\right]+\int_{-\infty}^{\infty}\frac{d\zeta}{2\pi}\frac{\alpha_{31}(\zeta)}{\alpha_{33}(\zeta)}e^{-i\zeta z}\ , \tag{2.18}$$

where prime denotes the derivative with respect to ζ.

Taking into account the integral representation for $\psi^{(3)}(x;\zeta)$ and using Eq. (2.3), we find the following expression for the "potential" $u(x)$:

$$u(x)=-2K_1^{(3)}(x,x)\ . \tag{2.19}$$

From Eqs. (2.17a)–(2.17c) we finally obtain the GLM equation for $K_1^{(3)}(x,y)$:

$$K_1^{(3)}(x,y)+F(x+y)+\int_x^\infty dz\, K_1^{(3)}(x,z)\int_x^\infty dx[F^*(z+s)F(s+y)+F(z+s)F^*(s+y)]=0\ . \tag{2.20}$$

In the following we find the time dependence of the scattering data. It should be noted that the time evolution equation (2.5) does not allow for the time-independent asymptotic condition (2.7) and (2.8). Due to this fact we introduce the time-dependent eigenfunctions which are defined as

$$\bar{\varphi}^{(i)}=\varphi^{(i)}\exp(\gamma_i V_- t)\ , \tag{2.21}$$

$$\bar{\psi}^{(i)}=\psi^{(i)}\exp(\gamma_i V_- t)\ , \tag{2.22}$$

where $V_-=\lim_{x\to-\infty}V_{33}(x)=4i\delta\zeta^3$, $i=1,2,3$.

By using the asymptotic form ($|x|\to\infty$) of Eq. (2.5), Eq. (2.10), and the time-dependent eigenfunctions (2.21) and (2.22), one can easily find the time dependence of the scattering data:

$$\begin{aligned}&\alpha_{33}(\zeta,t)=\alpha_{33}(\zeta,0),\quad \alpha_{ij}(\zeta,t)=\alpha_{ij}(\zeta,0);\ i,j=1,2\ ;\\ &\alpha_{3i}(\zeta,t)=\alpha_{31}(\zeta,0)\exp(-8i\delta\zeta^3 t),\quad \alpha_{i3}(\zeta,t)=\alpha_{i3}(\zeta,0)\exp(8i\delta\zeta^3 t)\ ,\\ &c_{3i}^{(j)}(t)=c_{3i}^{(j)}(0)\exp(-8i\delta\zeta_j^{*3}t);\ i=1,2\ .\end{aligned} \tag{2.23}$$

III. SOLITON SOLUTIONS

In the previous section we have given the location of the zeros of the $\alpha_{33}(\zeta)$, which is a basic criterion for classifying soliton solutions.

First we analyze case (i) when the diagonal element $\alpha_{33}(\zeta)$ of the scattering matrix has only one zero on the imaginary axis at $\zeta^*=-i\eta/2$, $\eta>0$. In addition we put $\alpha_{31}(\zeta)=0$ for real ζ.

In this case the function $F(z)$ is given by

$$F(z)=a(t)\exp\left[-\frac{\eta z}{2}\right]\ , \tag{3.1}$$

where $a(t)=ic_{31}(t)/\alpha_{33}'(-i\eta/2)$.

Considering that the function $K_1^{(3)}(x,y)$ has the form $K_1^{(3)}(x,y)=K(x)\exp(-\eta y/2)$ from Eqs. (2.19), (2.20), (2.23), and (3.1), we obtain the simplest single-soliton solution of Eq. (2.2):

$$u(x,t)=\frac{\eta}{\sqrt{2}}\operatorname{sech}[\eta(x-\delta\eta^2 t-x_0)]e^{i\varphi_0}\ , \tag{3.2}$$

where $x_0=(1/\eta)\ln\sqrt{2}|a(0)|/\eta$ and $\varphi_0=\arg a(0)$.

Thus we can write the one-parameter single-soliton solution of Eq. (1.2) in the form

$$q(Z,T)=\frac{\eta}{\sqrt{2}}\operatorname{sech}\left\{\eta\left[T-\left[\delta\eta^2+\frac{1}{12\delta}\right]Z-T_0\right]\right\}\exp\left\{i\left[\frac{1}{6\delta}\left[T-\frac{Z}{18\delta}\right]+\varphi_0\right]\right\}\ , \tag{3.3}$$

where $T_0 = x_0$.

We mention that this soliton solution was also obtained in [49] in a particular case [see Eqs. (38) and (51) in Ref. [49]].

Now we give some estimations for the threshold peak power below which the single soliton (3.3) cannot be formed. We notice that the single-soliton solution (3.2) is real regardless a constant complex phase. From this reason the condition of the absence of the discrete spectrum of the eigenvalue problem (2.3) with real u is (see, e.g., Ref. [28])

$$\sqrt{2}\int_{-\infty}^{\infty} u\, dx < 0.904 \ . \tag{3.4}$$

Supposing that the initial pulse shape is $u(x,0) = A\ \mathrm{sech}x$, from Eq. (3.4) one can obtain the threshold value of the amplitude A below which the single soliton (3.3) cannot be formed: $A_{\mathrm{th}} = 0.203$. The corresponding threshold peak power is

$$P_{\mathrm{th}} = 2A_{\mathrm{th}}^2 \frac{S_{\mathrm{eff}}\lambda^3|D|}{4\pi^2 c n_2 t_c^2} \ , \tag{3.5}$$

where S_{eff} is the effective core area, $D = (2\pi c/\lambda^2)(\partial^2 k/\partial\omega^2)$ is the group-velocity dispersion parameter, n_2 is the nonlinear refractive index coefficient, and t_c is the characteristic time of the pulse.

For a dispersion shifted fiber with typical parameters

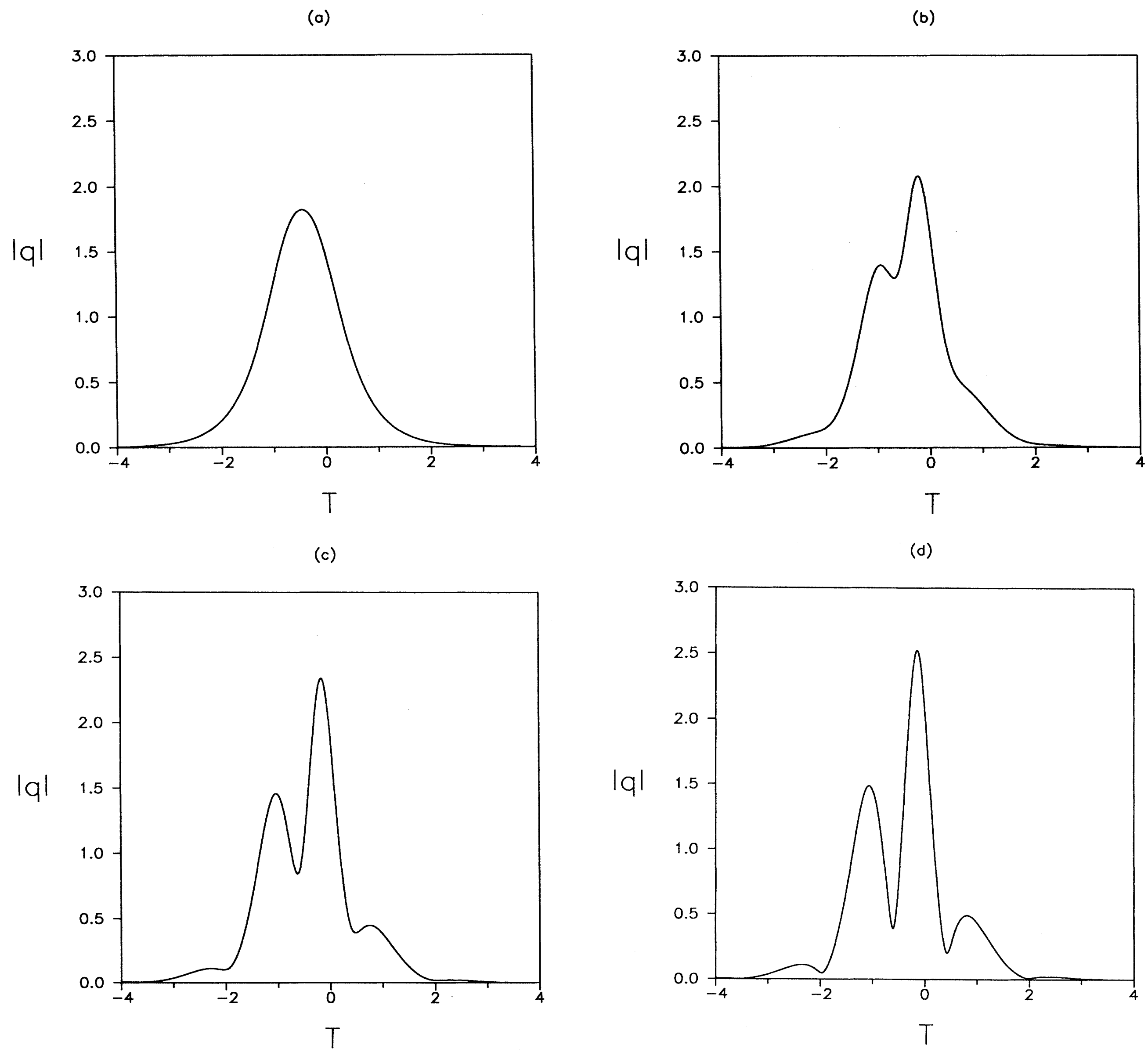

FIG. 1. The shape of $|q|$ as a function of T for the general single-soliton solution (3.8), with $\varphi_a = \pi/2$, $\varphi_b = 0$, $\xi = 2$, $\eta = 2$, $|a_0| = 1$, for (a) $r = 0$, (b) $r = 0.25$, (c) $r = 0.5$, and (d) $r = 0.75$.

$S_{\rm eff}=50$ μm^2, $|D|=0.4$ ps/nm km, $n_2=3.2\times10^{-20}$ m^2/W at $\lambda=1.55$ μm we obtain $P_{\rm th}\simeq 6.5$ W for $t_c=50$ fs ($\tau_{\rm FWHM}\simeq 88$ fs) and $P_{\rm th}\simeq 168.8$ W for $t_c=10$ fs ($\tau_{\rm FWHM}\simeq 17.6$ fm).

Next, we discuss case (ii) when the diagonal element of the scattering matrix $\alpha_{33}(\zeta)$ has two zeros $(\zeta^*, -\zeta)$ where $\zeta=(-\xi+i\eta)/2$ with $\xi,\eta>0$.

In order to find the general single-soliton solution in this case we consider that the function $K_1^{(3)}(x,y)$ has the following expression:

$$K_1^{(3)}(x,y)=L(x)e^{-i\zeta^* y}+M(x)e^{i\zeta y}\ . \tag{3.6}$$

As in the previous case we take $\alpha_{31}(\zeta)=0$ for real ζ so that the function $F(z)$ is

$$F(z)=a(t)e^{-i\zeta^* z}+b(t)e^{i\zeta z}\ , \tag{3.7}$$

where $a(t)=ic_{31}(t)/\alpha'_{33}(\zeta^*)$ and $b(t)=ic^*_{32}(t)/\alpha'_{33}(-\zeta)$. In this case the general single-soliton solution is

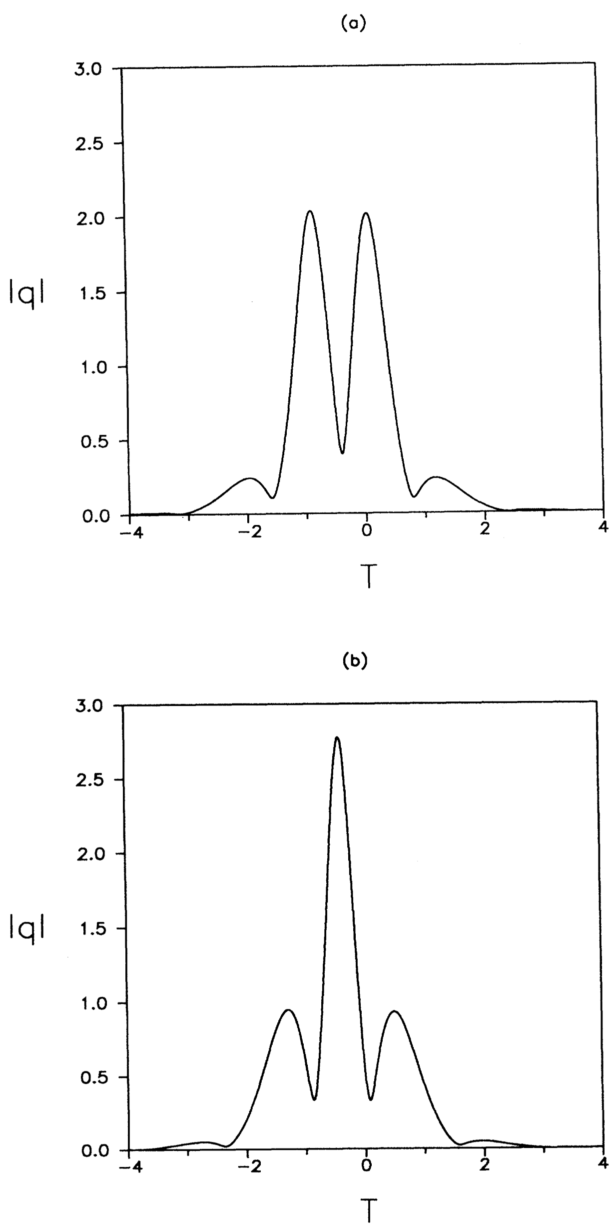

FIG. 2. The shape of $|q|$ as a function of T for the general single-soliton solution (3.8), with $r=0.75$, $\xi=2$, $\eta=2$, $|a_0|=1$, $\varphi_b=0$, for (a) $\varphi_a=0$ and (b) $\varphi_a=\pi$.

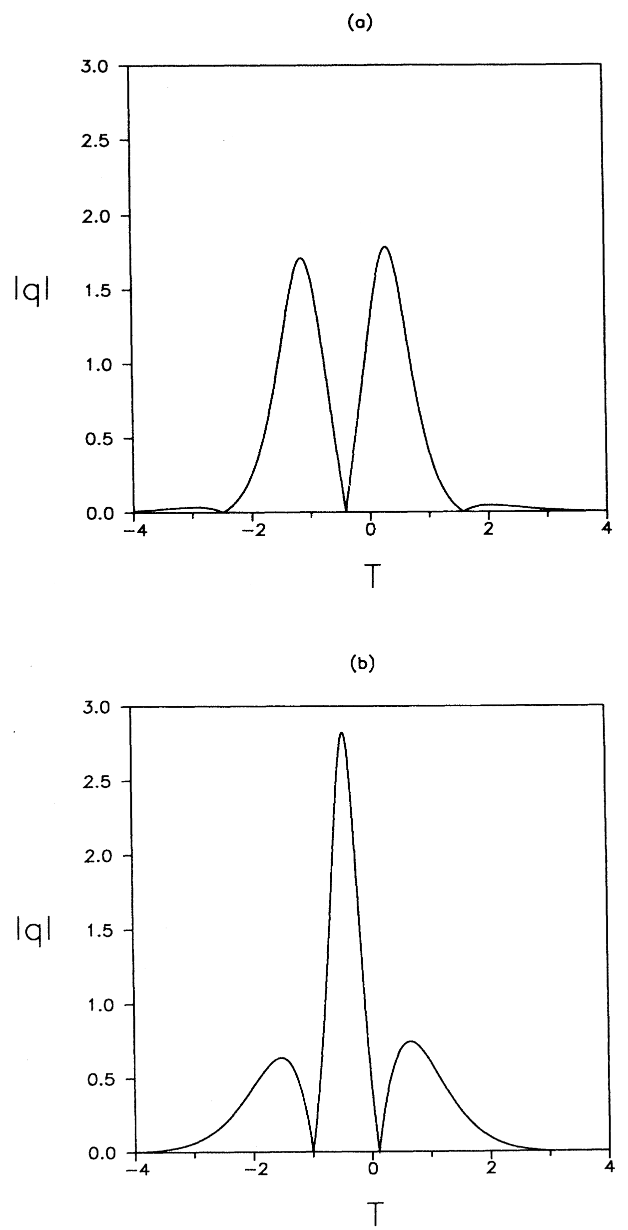

FIG. 3. The shape of $|q|$ as a function of T for the breather soliton solution (3.10), with $r=1$, $\xi=1$, $\eta=2$, $|a_0|=|b_0|=\sqrt{2}$, $\varphi_a=-\varphi_b$, for (a) $\varphi_a=0$ or $\varphi_a=\pi$, (b) $\varphi_a=\pi/2$ or $\varphi_a=3\pi/2$.

$$u(x,t)=-\frac{2e^{i(\varphi_a+\varphi_b)/2}}{\Delta}e^{-A}\left\{|a_0|\left[\frac{|a_0b_0|e^{-2(A+iB)}}{2\zeta^2}-\frac{(|a_0|^2+|b_0|^2)e^{-2A}}{\eta^2}-1\right]e^{iB}\right.$$
$$+|b_0|\left[\frac{|a_0b_0|e^{-2(A-iB)}}{\zeta^*}-\frac{(|a_0|^2+|b_0|^2)e^{-2A}}{2\zeta}\right]\frac{e^{-iB}}{i\eta}$$
$$+|b_0|\left[\frac{|a_0b_0|e^{-2(A-iB)}}{2\zeta^{*2}}-\frac{(|a_0|^2+|b_0|^2)e^{-2A}}{\eta^2}-1\right]e^{-iB}$$
$$\left.-|a_0|\left[\frac{|a_0b_0|e^{-2(A+iB)}}{\zeta}-\frac{(|a_0|^2+|b_0|^2)e^{-2A}}{2\zeta^*}\right]\frac{e^{iB}}{i\eta}\right\}, \tag{3.8}$$

where

$$\Delta=\left|\left[\frac{|a_0b_0|e^{-2(A-iB)}}{2\zeta^{*2}}-\frac{(|a_0|^2+|b_0|^2)e^{-2A}}{\eta^2}-1\right]\right|^2-\frac{1}{\eta^2}\left|\left[\frac{|a_0b_0|e^{-2(A-iB)}}{\zeta^*}-\frac{(|a_0|^2+|b_0|^2)e^{-2A}}{2\zeta}\right]\right|^2,$$

$A=\eta[x-\delta(\eta^2-3\xi^2)t]$, $B=\xi[x+\delta(\xi^2-3\eta^2)t]+(\varphi_a-\varphi_b)/2$, $a_0=a(0)$, $b_0=b(0)$, $\varphi_a=\arg a(0)$, and $\varphi_b=\arg b(0)$.

In order to obtain the breather soliton solution one can choose $|a_0|=|b_0|=\kappa$ and $\arg a_0=-\arg b_0$. With this choice the soliton solution is

$$u(x,t)=2\eta\frac{\kappa e^{-A}[\eta\cos(B+2\phi)+4|\zeta|\sin(B+\phi)]-4\sqrt{2}|\zeta|^2\cos B\cosh(A+\psi)}{\kappa^2e^{-2A}(1+\cos^2\phi)+\eta^2\cos[2(B+\phi)]-8|\zeta|^2\cosh^2(A+\psi)}, \tag{3.9}$$

where $\psi=\ln\eta/\sqrt{2}\kappa$ and $\phi=\arg\zeta$.

This solution described by four independent parameters $(\kappa,\xi,\eta,\varphi_a)$ represents a pulse moving with the velocity $\delta(\eta^2-3\xi^2)$ performing internal oscillations. Then the breatherlike soliton solution of Eq. (1.2) can be obtained from Eq. (3.9) via the transformation (2.1). In the limit $\xi\to 0$ the breather soliton (3.9) becomes the single-soliton solution (3.3) with T_0,φ_0 modified in view of our choice (3.7).

We mention that the breather soliton solution (3.9) can be put in the form

$$u(x,t)=\sqrt{2}\xi\eta\frac{\xi\cosh(A+\rho)\sin(B+\gamma)+\eta\sinh(A+\rho)\cos(B+\gamma)}{\xi^2\cosh^2(A+\rho)+\eta^2\cos^2(B+\gamma)}, \tag{3.10}$$

where $e^{\rho}=(2|\zeta|/\xi)e^{\psi}$, $\gamma=\phi-\pi/2$, and $\kappa e^{i\varphi_a}=a(0)$.

We notice that if one chooses $|a_0|=\kappa$, $|b_0|=0$ in Eq. (3.8), one can find the Sasa-Satsuma single-soliton solution [see Eqs. (38) and (39) in Ref. [49]]. We mention also that the evolution of this single soliton, under the action of a small dissipative term, or the term accounting for the intrapulse Raman scattering was thoroughly investigated in [53] by using the simplest technique of the perturbation theory based on the so-called balance equations for the quantities which are integrals of motion of the unperturbed equation [54].

A suitable parameter for describing the qualitative behavior of the single-soliton solution (3.8) is the ratio $r=|b_0|/|a_0|$. It is not necessary to analyze this solution for $r\in[0,\infty)$, but $r\in[0,1]$ because when $r\to 1/r$ the solution $u(x,t)\to u^*(x,t)$ (up to a phase factor). In Figs. 1(a)–1(d) it is shown the influence of the parameter r on the shape of the solution (3.8) written in (Z,T) variables. We mention that for this choice of the parameters the Sasa-Satsuma single-soliton solution [i.e., $r=0$ in (3.8)] has only one "hump" [see Fig. 1(a)]. Figures 2(a) and 2(b) present the drastic change of the shape of the same solution when the parameter φ_a changes from 0 to π. Figures 3(a) and 3(b) show the profile of the breather soliton (3.10) for different values of φ_a. Thus, we arrive at the conclusion that the general single-soliton solution (3.8) can have an arbitrary number of "humps" by an appropriate choice of its parameters.

Next we show the procedure to obtain the general N-soliton solution of Eq. (2.2). In this case we consider that $\alpha_{31}(\zeta)=0$ for real ζ and $\alpha_{33}(\zeta)$ has N pairs of zeros located at $(\zeta_i^*,-\zeta_i)$, $\zeta_i=(-\xi_i+i\eta_i)/2$, ξ_i, $\eta_i>0$, $i=1,2,\ldots,N$. With this choice the function $F(z)$ is

$$F(z)=\sum_{j=1}^{N}\left[a_j(t)\exp(-i\zeta_j^*z)+b_j(t)\exp(i\zeta_jz)\right], \tag{3.11}$$

where $a_i(t)=ic_{31}^{(i)}(t)/\alpha_{33}'(\zeta_i^*)$ and $b_i(t)=ic_{32}^{(i)*}(t)/\alpha_{33}'(-\zeta_i)$, $i=1,2,\ldots,N$.

By choosing $K_1^{(3)}(x,y)$ in the form

$$K_1^{(3)}(x,y)=\sum_{j=1}^{N}[L_j(x)\exp(-i\zeta_j^*y)+M_j(x)\exp(i\zeta_jY)], \tag{3.12}$$

we obtain from the GLM equation (2.20) a system of linear algebraic equations for $L_j(x)$ and $M_j(x)$, $j=1,2,\ldots,N$. By solving this system one can obtain

$$u(x,t) = -2\sum_{j=1}^{N}\left[\exp(-i\zeta_j^* x)\frac{\det A^{(j)}}{\det A} + \exp(i\zeta_j x)\frac{\det A^{(j+N)}}{\det A}\right], \quad (3.13)$$

where A is a $2N\times 2N$ matrix of the form

$$A = \begin{bmatrix} P & Q \\ Q^* & P^* \end{bmatrix} \quad (3.14)$$

and P, Q are $N\times N$ matrices with the following elements:

$$P_{ij} = \sum_{m=1}^{N}\left[\frac{(a_i a_m^* + b_m b_i^*)\exp[i(2\zeta_m - \zeta_i^* - \zeta_j^*)x]}{(\zeta_m - \zeta_i^*)(\zeta_m - \zeta_j^*)} + \frac{(a_i b_m^* + a_m b_i^*)\exp[-i(2\zeta_m^* + \zeta_i^* + \zeta_j^*)x]}{(\zeta_m^* + \zeta_i^*)(\zeta_m^* + \zeta_j^*)}\right] - \delta_{ij}, \quad (3.15a)$$

$$Q_{ij} = \sum_{m=1}^{N}\left[\frac{(a_i a_m^* + b_m b_i^*)\exp[i(2\zeta_m - \zeta_i^* + \zeta_j)x]}{(\zeta_m - \zeta_i^*)(\zeta_m + \zeta_j)} + \frac{(a_i b_m^* + a_m b_i^*)\exp[-i(2\zeta_m^* + \zeta_i^* - \zeta_j)x]}{(\zeta_m^* + \zeta_i^*)(\zeta_m^* - \zeta_j)}\right], \quad (3.15b)$$

for $i,j = 1,2,\dots,N$.

Here the matrix $A^{(i)}$ is obtained by replacing the ith column of the matrix A with $[a_1\exp(-i\zeta_1^* x), \dots, a_N\exp(-i\zeta_N^* x), b_1\exp(i\zeta_1 x), \dots, b_N\exp(i\zeta_N x)]^t$.

Finally, the N-soliton solution of Eq. (1.2) is obtained from Eq. (3.13) by using the transformation (2.1).

For Eq. (2.2) we derive several conserved quantities in Appendix B.

IV. CONCLUSIONS AND REMARKS

Using the IST, femtosecond solitons in optical fibers have been found for an appropriate choice of the fiber parameters. The soliton solutions have been classified by the positions of the zeros of the diagonal element $\alpha_{33}(\zeta)$ of the scattering matrix α. For the one-parameter single-soliton solution an estimation of the threshold power below which the soliton cannot be formed was given. In addition, the four-parameter breather soliton solution and the general N-soliton solution were also presented. Unlike the NLSE, where a single-soliton solution corresponds to only one zero of the diagonal element of the scattering matrix, for our perturbed NLSE the single-soliton solution corresponds to a pair of zeros of the diagonal matrix element $\alpha_{33}(\zeta)$ located symmetrically with respect to the imaginary axis in the lower complex half plane ζ. The general solution is very rich in showing qualitatively different behaviors. The classification is made against the parameters $|b_0|/|a_0|$, φ_a, and φ_b, which are responsible for the multifarious nature of the single-soliton solution.

Finally, we stress that Eq. (1.2) holds when the ratios among the coefficients of the higher-order terms β_1:β_2:β_3 in the perturbed NLSE amount to 1:6:3. Hence, by properly tailoring the optical fiber this situation can be realized in the femtosecond regime [24] and we expect that the simplest single-soliton could be observed experimentally.

ACKNOWLEDGMENT

One of the authors (D.M.) acknowledges financial support from the Spanish Ministry of Education and Science.

APPENDIX A: ANALYTICITY PROPERTIES OF JOST FUNCTIONS AND SCATTERING MATRIX ELEMENTS

In this Appendix we show in detail the way to establish the analyticity of one of the Jost functions, namely, $\chi^{(1)}(x;\zeta) \equiv \varphi^{(1)}(x;\zeta)e^{i\zeta x}$. From (2.3) we easily find that the function $\chi^{(1)}(x;\zeta)$ obeys the following equation:

$$\frac{\partial\chi^{(1)}}{\partial x} = 2i\zeta\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\chi^{(1)} + \tilde{U}\chi^{(1)}, \quad (A1)$$

where

$$\tilde{U} = \begin{bmatrix} 0 & 0 & u \\ 0 & 0 & u^* \\ -u^* & -u & 0 \end{bmatrix} \quad (A2)$$

is the "potential" matrix of the IST problem.

From (A1) we have the following integral equation for $\chi^{(1)}$:

$$\chi^{(1)}(x;\zeta) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \int_{-\infty}^{x} dy \begin{bmatrix} (\tilde{U}(y)\chi^{(1)}(y))_1 \\ (\tilde{U}(y)\chi^{(1)}(y))_2 \\ e^{2i\zeta(x-y)}(\tilde{U}(y)\chi^{(1)}(y))_3 \end{bmatrix}. \quad (A3)$$

Under suitable conditions, we can extend $\chi^{(1)}(x;\zeta)$ into the upper half of the complex ζ plane. To see this, let A be the matrix:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \quad (A4)$$

For ζ in the upper complex half plane, from Eq. (A3) it is easy to obtain that $|\chi^{(1)}(x;\zeta)|$ is bounded by the following series:

$$|\chi^{(1)}(x;\zeta)| \le \left[\sum_{n\ge 0}\frac{1}{n!}M^n(x)A^n\right]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad (A5)$$

where $M(x) = \int_{-\infty}^{x} |u(y)|dy$.

Taking into account that $A^{2n}=2^{n-1}X$ and $A^{2n+1}=2^nA$, where

$$X=\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}, \tag{A6}$$

we have that for ζ in the upper complex half plane, for real x there exists $|\chi^{(1)}(x;\zeta)|$ is there exists the following integral:

$$M(\infty)=\int_{-\infty}^{\infty}|u(x)|dx . \tag{A7}$$

In a similar manner one can show that the functions $\chi^{(2)}(x;\zeta)\equiv\varphi^{(2)}(x;\zeta)e^{i\zeta x}$ and $\Omega^{(3)}(x;\zeta)\equiv\psi^{(3)}(x;\zeta)e^{-i\zeta x}$ can be analytically continued in the upper complex half plane ζ, while $\Omega^{(1)}(x;\zeta)\equiv\psi^{(1)}(x;\zeta)e^{i\zeta x}$, $\Omega^{(2)}(x;\zeta)\equiv\psi^{(2)}(x;\zeta)e^{i\zeta x}$, and $\chi^{(3)}(x;\zeta)\equiv\varphi^{(3)}(x;\zeta)e^{-i\zeta x}$ can be analytically continued in the lower complex half plane ζ.

Next we establish the analyticity properties of the scattering matrix elements $\alpha_{ij}(\zeta)$. From Eq. (2.10) we have

$$\alpha_{ij}(\zeta)=\lim_{x\to\infty}\varphi_j^{(i)}(x;\zeta)e^{-i\gamma_j\zeta x} . \tag{A8}$$

Taking into account the analyticity properties of the functions $\chi^{(i)}(x;\zeta)$ and $\Omega^{(i)}(x;\zeta)$, $i=1,2,3$ established above it results that $\alpha_{11}(\zeta)$, $\alpha_{12}(\zeta)$, $\alpha_{21}(\zeta)$, $\alpha_{22}(\zeta)$, and $\alpha_{33}^*(\zeta^*)$ can be analytically continued in the upper complex half plane ζ and $\alpha_{11}^*(\zeta^*)$, $\alpha_{12}^*(\zeta^*)$, $\alpha_{21}^*(\zeta^*)$, $\alpha_{22}^*(\zeta^*)$, and $\alpha_{33}(\zeta)$ can be analytically continued in the lower complex half plane ζ if $|u|$ tends to zero sufficiently fast as $|x|\to\infty$ such that there exists the integral (A7).

APPENDIX B: INTEGRALS OF MOTION

From Eqs. (2.8) and (2.10) one can see that some of the elements of the scattering matrix are time independent. Expanding these time-independent matrix elements in power series with respect to ζ^{-1} we can find the infinite set of conservation laws. In the following we will show in detail the procedure of obtaining the conserved quantities related to the power-series expansion of the diagonal matrix element $\alpha_{33}(\zeta)$.

From Eqs. (2.8) and (2.10) it results that

$$\alpha_{33}(\zeta)=\lim_{x\to\infty}\chi_3^{(3)}(x;\zeta) . \tag{B1}$$

It is a simple matter to show that the components of the function $\chi^{(3)}(x;\zeta)$ can be expanded in power series with respect to ζ^{-1} as follows:

$$\chi_1^{(3)}(x;\zeta)=\sum_{n\geq 1}\frac{A_1^{(n)}}{\zeta^n} , \tag{B2a}$$

$$\chi_2^{(3)}(x;\zeta)=\sum_{n\geq 1}\frac{A_2^{(n)}}{\zeta^n} , \tag{B2b}$$

$$\chi_3^{(3)}(x;\zeta)=1+\sum_{n\geq 1}\frac{A_3^{(n)}}{\zeta^n} . \tag{B2c}$$

From Eqs. (B1) and (B2c) it results that the conserved quantities are given by

$$I_3^{(n)}=\int_{-\infty}^{\infty}A_3^{(n)}dx . \tag{B3}$$

The functions $\chi_i^{(3)}(x;\zeta)$, $i=1,2,3$, obey the following system:

$$\begin{aligned}\frac{\partial\chi_1^{(3)}}{\partial x}&=-2i\zeta\chi_1^{(3)}+u\chi_3^{(3)} ,\\ \frac{\partial\chi_2^{(3)}}{\partial x}&=-2i\zeta\chi_2^{(3)}+u^*\chi_3^{(3)} ,\\ \frac{\partial\chi_3^{(3)}}{\partial x}&=-u^*\chi_1^{(3)}-u\chi_2^{(3)} .\end{aligned} \tag{B4}$$

Next, taking into account the relations (B2a)–(B2c), the system (B4) becomes

$$\begin{aligned}\frac{\partial A_1^{(n)}}{\partial x}&=-2iA_1^{(n+1)}+uA_3^{(n)} ,\\ \frac{\partial A_2^{(n)}}{\partial x}&=2iA_2^{(n+1)}+u^*A_3^{(n)} ,\\ \frac{\partial A_3^{(n)}}{\partial x}&=-u^*A_1^{(n)}-uA_2^{(n)} ,\end{aligned} \tag{B5}$$

with

$$A_1^{(1)}=-\frac{i}{2}u,\quad A_2^{(1)}=-\frac{i}{2}u^* . \tag{B6}$$

From the system (B5), with the relations (B6) one can successively determine the unknown functions $A_i^{(n)}$ ($i=1,2,3$). Thus the first three conserved quantities are given by

$$I_3^{(1)}=i\int_{-\infty}^{\infty}dx|u|^2 , \tag{B7a}$$

$$I_3^{(2)}=-\frac{1}{2}\left[\int_{-\infty}^{\infty}dx|u|^2\right]^2 , \tag{B7b}$$

$$I_3^{(3)}=\frac{1}{2i}\left[\int_{-\infty}^{\infty}dx|u|^4-\frac{1}{2}\int_{-\infty}^{\infty}dx\left[\frac{du}{dx}\right]^2+\frac{1}{6}\left[\int_{-\infty}^{\infty}dx|u|^2\right]^3\right] . \tag{B7c}$$

In a similar manner we obtain the conservation laws related to the power-series expansions of the diagonal matrix elements $\alpha_{22}(\zeta)$ and $\alpha_{11}(\zeta)$:

$$I_2^{(1)}=\frac{1}{2i}\int_{-\infty}^{\infty}dx|u|^2 , \tag{B8a}$$

$$I_2^{(2)}=-\frac{1}{4}\left[\int_{-\infty}^{\infty}dx\,u^*\frac{du}{dx}+\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,u^{*2}(x)u^2(y)+\frac{1}{2}\left[\int_{-\infty}^{\infty}dx|u|^2\right]^2\right] , \tag{B8b}$$

$$I_2^{(3)}=\frac{i}{8}\left[-\int_{-\infty}^{\infty}dx\left[\frac{du}{dx}\right]^2+2\int_{-\infty}^{\infty}dx|u|^4+\left[\int_{-\infty}^{\infty}dx\,u^*\frac{du}{dx}\right]\left[\int_{-\infty}^{\infty}dx|u|^2\right]\right.$$
$$+\frac{1}{6}\left[\int_{-\infty}^{\infty}dx|u|^2\right]^3+\int_{-\infty}^{\infty}dx\int_{-\infty}^{x}dy\int_{-\infty}^{y}dz|u(x)|^2u^{*2}(y)u^2(z)$$
$$\left.+\int_{-\infty}^{\infty}dx\,u^{*2}(x)\left[\int_{-\infty}^{x}dy\,u^2(y)\right]\left[\int_{-\infty}^{x}dy|u(y)|^2\right]\right], \tag{B8c}$$

$$I_1^{(1)}=-I_2^{(1)*},\quad I_1^{(2)}=I_2^{(2)*},\quad I_1^{(3)}=-I_2^{(3)*}\ . \tag{B8d}$$

We note that the relations (B8d) can be obtained also from the symmetry properties (2.14).

---

*Permanent address: Institute of Atomic Physics, Department of Theoretical Physics, P.O. Box MG-6, Bucharest, Romania.

[1] A. Hasegawa and F. Tappert, Appl. Phys. Lett. **23**, 142 (1973); **23**, 171 (1973).

[2] L. F. Mollenauer, R. H. Stolen, and J. P. Gordon, Phys. Rev. Lett. **45**, 1095 (1980).

[3] K. J. Blow and N. J. Doran, Phys. Lett. **107A**, 55 (1985); W. J. Tomlinson, R. J. Hawkins, A. M. Weiner, J. P. Heritage, and R. N. Thurston, J. Opt. Soc. Am. B **6**, 329 (1989).

[4] P. Emplit, J. P. Hamaide, F. Reynaud, C. Froehly, and A. Barthelemy, Opt. Commun. **62**, 374 (1987); D. Krökel, N. J. Halas, G. Giuliani, and D. Grischkowski, Phys. Rev. Lett. **60**, 29 (1988); A. M. Weiner, J. P. Heritage, R. J. Hawkins, R. N. Thurston, E. M. Kirschner, D. E. Leaird, and W. J. Tomlinson, *ibid.* **61**, 2445 (1988).

[5] Y. Kodama and A. Hasegawa, Opt. Lett. **7**, 339 (1982).

[6] H. G. Winful, Appl. Phys. Lett. **46**, 527 (1985).

[7] D. Anderson and M. Lisak, Phys. Rev. A **32**, 3270 (1985).

[8] D. N. Christodoulides and R. I. Joseph, Appl. Phys. Lett. **47**, 76 (1985).

[9] M. J. Potasek, G. P. Agrawal, and S. C. Pinault, J. Opt. Soc. Am. B **3**, 205 (1986).

[10] J. P. Gordon, Opt. Lett. **11**, 662 (1986); F. M. Mitschke and L. F. Mollenauer, *ibid.* **11**, 659 (1986).

[11] M. J. Potasek and G. P. Agrawal, Phys. Rev. A **36**, 3862 (1987).

[12] P. K. A. Wai, C. R. Menyuk, H. H. Chen, and Y. C. Lee, Opt. Lett. **12**, 628 (1987); IEEE J. Quantum Electron. **QE-24**, 373 (1988).

[13] B. Crosignani, A. Yariv, and P. Di Porto, Opt. Commun. **65**, 387 (1988).

[14] A. D. Boardman and G. S. Cooper, J. Opt. Soc. Am. B **5**, 403 (1988).

[15] S. Trillo, S. Wabnitz, E. M. Wright, and G. I. Stegeman, Opt. Lett. **13**, 672 (1988).

[16] K. J. Blow, N. J. Doran, and D. Wood, Opt. Lett. **12**, 202 (1987).

[17] E. M. Dianov, P. V. Mamyshev, A. M. Prokhorov, and S. V. Chernikov, Opt. Lett. **14**, 1008 (1989); S. V. Chernikov and P. V. Mamyshev, J. Opt. Soc. Am. B **8**, 1633 (1991).

[18] A. B. Aceves and S. Wabnitz, Phys. Lett. A **141**, 37 (1989).

[19] C. M. De Sterke and J. E. Sipe, Opt. Lett. **14**, 871 (1989).

[20] Y. Lai and H. Haus, Phys. Rev. A **40**, 844 (1989); **40**, 854 (1989).

[21] Yu. S. Kivshar and B. A. Malomed, Opt. Lett. **14**, 1365 (1989).

[22] K. Hayata, K. Saka, and M. Koshiba, J. Appl. Phys. **68**, 4971 (1990).

[23] Y. Kodama, J. Stat. Phys. **39**, 597 (1985); Y. Kodama and A. Hasegawa, IEEE J. Quantum Electron. **QE-23**, 510 (1987).

[24] M. J. Potasek, J. Appl. Phys. **65**, 941 (1989); M. J. Potasek and M. Tabor, Phys. Lett. A **154**, 449 (1991); M. J. Potasek, IEEE J. Quantum Electron. **QE-29**, 281 (1993).

[25] F. Singer, M. J. Potasek, J. M. Fang, and M. C. Teich, Phys. Rev. A **46**, 4192 (1992); F. Singer, M. J. Potasek, and M. C. Teich, Quantum Opt. **4**, 157 (1992).

[26] V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys. JETP **34**, 62 (1972)]; **64**, 1627 (1973) [**37**, 823 (1973)].

[27] J. Satsuma and N. Yajima, Prog. Theor. Phys. Suppl. **55**, 284 (1974).

[28] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Stud. Appl. Math. **53**, 249 (1974).

[29] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Method* (SIAM, Philadelphia, 1981).

[30] F. Calogero and A. Degasperis, *Spectral Transform and Solitons,* (North-Holland, Amsterdam, 1988).

[31] L. Gagnon and P. Winternitz, J. Phys. A **21**, 1493 (1988); **22**, 469 (1989); M. Florjanczyk and L. Gagnon, Phys. Rev. A **41**, 4478 (1990).

[32] N. N. Akhmediev, V. M. Eleonskii, and N. E. Kulagin, Teor. Mat. Fiz. **72**, 183 (1987) [Theor. Math. Phys. **72**, 809 (1987)].

[33] D. Mihalache and N. C. Panoiu, Phys. Rev. A **45**, 6730 (1992); J. Math. Phys. **33**, 2323 (1992).

[34] D. Mihalache, F. Lederer, and D.-M. Baboiu, Phys. Rev. A **47**, 3285 (1993); N. Akhmediev and A. Ankiewicz, *ibid.* **47**, 3213 (1993); L. Gagnon, J. Opt. Soc. Am. B **10**, 469 (1993).

[35] D. J. Kaup, SIAM J. Appl. Math. **31**, 121 (1976); Phys. Rev. A **42**, 5689 (1990); **44**, 4582 (1991).

[36] G. P. Agrawal, *Nonlinear Fiber Optics* (Academic, Boston, 1989).

[37] N. Tzoar and M. Jain, Phys. Rev. A **23**, 1266 (1979).

[38] D. Anderson and M. Lisak, Phys. Rev. A **27**, 1393 (1983).

[39] E. Bourkoff, W. Zhao, R. I. Joseph, and D. N. Christodoulides, Opt. Lett. **12**, 272 (1987).

[40] K. Ohkuma, Y. H. Ichikawa, and Y. Abe, Opt. Lett. **12**, 516 (1987).

[41] V. N. Serkin, Pis'ma Zh. Tekh. Fiz. **13**, 772 (1987) [Sov. Phys. Tech. Phys. Lett. **13**, 320 (1987)].

[42] A. B. Grudinin, V. N. Men'shov, and T. N. Fursa, Zh. Eksp. Teor. Fiz. **97**, 449 (1990) [Sov. Phys. JETP **70**, 249 (1990)].
[43] G. P. Agrawal and C. Headley III, Phys. Rev. A **46**, 1573 (1992).
[44] J. R. de Oliveira, M. A. de Moura, J. M. Hickmann, and A. S. L. Gomes, J. Opt. Soc. Am. B **9**, 2025 (1992).
[45] D. Mihalache, N. Truta, N.-C. Panoiu, and D.-M. Baboiu, Phys. Rev. A **47**, 3190 (1993).
[46] D. J. Kaup and A. C. Newell, J. Math. Phys. **19**, 798 (1978).
[47] H. H. Chen, Y. C. Lee, and C. S. Liu, Phys. Scr. **20**, 490 (1979).
[48] R. Hirota, J. Math. Phys. **14**, 805 (1973).
[49] N. Sasa and J. Satsuma, J. Phys. Soc. Jpn. **60**, 409 (1991).
[50] V. E. Zakharov and S. V. Manakov, Zh. Eksp. Teor. Fiz. **69**, 1654 (1975) [Sov. Phys. JETP **42**, 842 (1976)]; S. V. Manakov, *ibid.* **65**, 505 (1973) [**38**, 248 (1974)].
[51] D. J. Kaup, Stud. Appl. Math. **55**, 9 (1976).
[52] N. Yajima and M. Oikawa, Prog. Theor. Phys. **56**, 1719 (1976).
[53] B. A. Malomed, N. Sasa, and J. Satsuma, Chaos, Solitons Fractals **1**, 383 (1991).
[54] Yu. S. Kivshar and B. A. Malomed, Rev. Mod. Phys. **61**, 763 (1989).